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Efficient design sensitivities of structures subjected to dynamic loading ☆

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Abstract

Calculation of design sensitivities often involves much computational effort, particularly in large structural systems with many design variables. Approximation concepts, which are often used to reduce the computational cost involved in repeated analysis, are usually not sufficiently accurate for sensitivity analysis. In this study, approximate reanalysis is used to improve the efficiency of dynamic sensitivity analysis. Using modal analysis, the response derivatives with respect to design variables are presented as a combination of sensitivities of the eigenvectors and the generalized displacements. A procedure intended to reduce the number of differential equations that must be solved during the solution process is proposed. Efficient evaluation of the derivatives, using finite difference and the recently developed combined approximations approach, is presented. Numerical examples show that high accuracy of design sensitivities can be achieved efficiently.

Keywords: Combined approximations; Design sensitivity; Dynamic analysis; Finite differences

1. Introduction

Design sensitivity analysis of structures deals with the calculation of the response derivatives with respect to the design variables. These derivatives, called the sensitivity coefficients, are used in the solution of various problems. In design optimization, the sensitivity coefficients are often required to select a search direction. These coefficients are used in generating approximations for the response of a modified system, including approximate reanalysis models and explicit approximations of the constraint functions in terms

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of the structural parameters. In addition, the sensitivities are required for assessing the effects of uncertainties in the structural properties on the system response. Calculation of the sensitivities often involves much computational effort, particularly in large structural systems with many design variables. As a result, there has been much interest in efficient procedures for calculating the sensitivity coefficients. Early and recent developments in methods for sensitivity analysis are discussed in many studies (e.g. Haug et al., 1986; Haftka and Adelman, 1989; Haftka and Gurdal, 1993; van Keulen et al., in press). Methods of sensitivity analysis for discretized systems can be divided into the following classes:

(a) Finite-difference methods, which are easy to implement but might involve numerous repeated analyses and high computational cost, particularly in problems with many design or response variables. In addition, finite-difference approximations might have accuracy problems. The efficiency can be improved by using fast reanalysis techniques.
(b) Analytical methods, which provide exact solutions but might not be easy to implement in some problems such as shape optimization.
(c) "Semi-analytical" methods, which are based on a compromise between finite-difference methods and analytical methods. These methods use finite-difference evaluation of the right-hand-side vector. They are easy to implement but might provide inaccurate results.

In general, the following factors are considered in choosing a suitable sensitivity analysis method for a specific application:

- The accuracy of the calculations.
- The computational effort involved.
- The ease-of-implementation.

The implementation effort is weighted against the performance of the algorithms as reflected in their computational efficiency and accuracy. The quality of the results and efficiency of the calculations are usually two conflicting factors. That is, higher accuracy is often achieved at the expense of more computational effort.

Dynamic sensitivity analysis has been demonstrated by several authors. Using the mode superposition approach and assuming harmonic loading, the response sensitivities were evaluated by direct differentiation of the equations of motion in the generalized coordinates (Kramer and Grierson, 1989). In cases of earthquake loading the ground acceleration is usually given in discrete time steps, thus the loading is not given analytically. In several studies (Kim and Choi, 2000; van Keulen et al., in press) the unconditionally stable implicit numerical equation was directly derived. It was found that the analysis equations and the sensitivity equations have the same left-hand side expression. Thus, it was possible to use the available factorized coefficient matrix. A numerical procedure was applied for calculation of the sensitivity of the response.

Approximation concepts are often used to reduce the computational cost involved in repeated analysis of structures (Kirsch, 2002). However, most approximations that are adequate for structural reanalysis are not sufficiently accurate for sensitivity analysis. In this study, approximate reanalysis is used to improve the efficiency of dynamic sensitivity analysis by finite differences. Given the results of exact analysis for an initial design, the displacements for various modified designs are evaluated efficiently by the recently developed combined approximations (CA) approach (Kirsch, 2002, 2003a). Originally, the approach was developed for linear static problems. Recently, accurate results were reported also for eigenproblem (Kirsch, 2003b; Kirsch and Bogomolni, 2004) and dynamic reanalysis problems (Kirsch et al., submitted for publication-a, submitted for publication-b).

Calculation of analytical derivatives using approximate analysis models have been demonstrated previously (Kirsch, 1994; Kirsch and Papalambros, 2001). It was found that accurate results can be achieved but, as noted earlier, analytical derivatives might not be easy to implement. It was demonstrated recently

(Kirsch et al., 2005) that accurate derivatives can be achieved efficiently by the CA approach and finite differences for linear static problems and eigenproblems.

The present study deals with the design sensitivity analysis for discrete linear systems subjected to dynamic loading. The problem of dynamic analysis by mode superposition is first introduced, and the response derivatives with respect to design variables are presented as a combination of sensitivities of the eigenvectors and the generalized displacements. A procedure for reducing the number of differential equations that must be solved during the solution process is then proposed. Computational procedures intended to improve the accuracy of the approximations are developed, and efficient evaluation of the response derivatives by the combined approximations approach is presented. Numerical examples demonstrate the accuracy of the results.

2. Problem formulation

2.1. Dynamic analysis

Consider the equations of motion for a linear system subjected to dynamic forces

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{R} \tag{1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{r}$ is the unknown displacement vector, and $\mathbf{R}$ is the load vector.

Considering mode superposition, we use the following transformation from the nodal displacements to the generalized displacements:

$$\mathbf{r} = \sum_{k=1}^{p} \mathbf{\Phi}_k Z_k = \mathbf{\Phi}\mathbf{Z} \tag{2}$$

where p is the number of mode shapes considered (in general $p \ll m$, where m is the number of degrees of freedom), $\mathbf{Z}$ is a vector of generalized displacements, and $\mathbf{\Phi}$ is the matrix of eigenvectors (mode shapes). The eigenvectors $\mathbf{\Phi}_k$ and eigenvalues $\lambda_k = \omega_k^2$ (ω_k are the circular frequencies) are obtained by solving the eigenproblem

$$\mathbf{K}\mathbf{\Phi}_k = \lambda_k \mathbf{M}\mathbf{\Phi}_k \quad k = 1, \ldots, p \tag{3}$$

In the presentation that follows we assume damping such that classical modal analysis can be used. Substituting Eq. (2) into Eq. (1) and pre-multiplying the resulting equations by $\mathbf{\Phi}^{\mathrm{T}}$, we obtain the uncoupled equations of motion

$$\mathbf{I}\ddot{\mathbf{Z}} + \mathbf{\Lambda}\dot{\mathbf{Z}} + \mathbf{\Omega}^2\mathbf{Z} = \mathbf{P} \tag{4}$$

In these equations the right-hand side vector in normalized coordinates is $\mathbf{P} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{R}$, the mass matrix is an identity matrix $\mathbf{I} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}$, the damping matrix is $\mathbf{\Lambda} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi}$, and the stiffness matrix is $\mathbf{\Omega}^2 = \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi}$. Note that $\mathbf{\Lambda}$ and $\mathbf{\Omega}^2$ in these coordinates are diagonal low-order matrices, given by

$$\begin{aligned} \mathbf{\Lambda} &= \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi} = \begin{bmatrix} 2\omega_1\zeta_1 & & \\ & \ddots & \\ & & 2\omega_p\zeta_p \end{bmatrix} \\ \mathbf{\Omega}^2 &= \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_p^2 \end{bmatrix} \end{aligned} \tag{5}$$

where ζ_k are the damping ratios, usually estimated by experimental data. Thus, Eq. (4) consists of the p uncoupled equations

$$\ddot{Z}_k + 2\omega_k\zeta_k\dot{Z}_k + \omega_k^2 Z_k = P_k \quad k = 1,\ldots,p \tag{6}$$

It should be noted that in many problems (e.g. earthquake loading) the load vector $\mathbf{R}$, and therefore the right-hand side terms

$$P_k = \mathbf{\Phi}_k^{\mathrm{T}}\mathbf{R} \tag{7}$$

are given as discrete values at each time step, and not by analytical functions.

In summary, computation of the dynamic response by modal analysis involves the following steps.

(a) Determine the matrices $\mathbf{K}$, $\mathbf{M}$, and $\mathbf{C}$.
(b) Determine the p requested eigenpairs λ_k, $\mathbf{\Phi}_k$ by solving the eigenproblem of Eq. (3).
(c) Compute the modal coordinates Z_k by solving Eq. (6).
(d) Compute the nodal displacements $\mathbf{r}$ by Eq. (2).
(e) Calculate the element forces using the element stiffness properties.

2.2. Displacement derivatives

The derivative expressions of the displacement vector $\mathbf{r}$ with respect to a design variable X_j, $\partial\,\mathbf{r}/\partial X_j$, are given by differentiating Eq. (2)

$$\frac{\partial \mathbf{r}}{\partial X_j} = \sum_{k=1}^{p}\left(\frac{\partial \mathbf{\Phi}_k}{\partial X_j}Z_k + \mathbf{\Phi}_k\frac{\partial Z_k}{\partial X_j}\right) \tag{8}$$

The derivatives $\partial\mathbf{\Phi}_k/\partial X_j$ can be evaluated efficiently by finite differences using the CA approach, as will be shown later. Assuming that the damping ratios ζ_k are independent of the design variables (which is typical, for example, in civil engineering structures), we calculate $\partial Z_k/\partial X_j$ by differentiation of Eq. (6)

$$\frac{\partial \ddot{Z}_k}{\partial X_j} + 2\omega_k\zeta_k\frac{\partial \dot{Z}_k}{\partial X_j} + \omega_k^2\frac{\partial Z_k}{\partial X_j} = \frac{\partial P_k}{\partial X_j} - 2\frac{\partial \omega_k}{\partial X_j}\zeta_k\dot{Z}_k - \frac{\partial \omega_k^2}{\partial X_j}Z_k \tag{9}$$

Denoting

$$q_k = \partial Z_k/\partial X_j \quad \dot{q}_k = \partial \dot{Z}_k/\partial X_j \quad \ddot{q}_k = \partial \ddot{Z}_k/\partial X_j \tag{10}$$

and substituting Eqs. (7) and (10) into Eq. (9) yields

$$\ddot{q}_k + 2\omega_k\zeta_k\dot{q}_k + \omega_k^2 q_k = \mathbf{\Phi}_k^{\mathrm{T}}\frac{\partial \mathbf{R}}{\partial X_j} + \frac{\partial \mathbf{\Phi}_k^{\mathrm{T}}}{\partial X_j}\mathbf{R} - 2\frac{\partial \omega_k}{\partial X_j}\zeta_k\dot{Z}_k - \frac{\partial \omega_k^2}{\partial X_j}Z_k \tag{11}$$

Note that the left-hand sides and the initial conditions of Eqs. (6) and (11) are similar (e.g. $q_k = \dot{q}_k = 0$ for $t = 0$), whereas the right-hand sides are different. This similarity will be used to reduce the number of differential equations that must be solved during the solution process.

In summary, given the eigenpairs and the response for a certain design and time, evaluation of the displacement derivatives involves the following steps.

(a) Evaluate the derivatives of the eigenpairs ($\partial\mathbf{\Phi}_k/\partial X_j$ and $\partial\lambda_k/\partial X_j$).
(b) Compute the right side of Eq. (11).
(c) Compute the derivatives $q_k = \partial Z_k/\partial X_j$ by solving Eq. (11).
(d) Evaluate the displacement derivatives $\partial\mathbf{r}/\partial X_j$ by Eq. (8).

Assuming a problem with p mode shapes and n design variables, the main computational effort is involved in the following two steps:

(a) Solution of the pn differential equation (11).
(b) Evaluation of pn derivatives of the eigenpairs ($\partial\mathbf{\Phi}_k/\partial X_j$ and $\partial\omega_k/\partial X_j$).

A procedure intended to reduce the number of differential equations to be solved during the solution process is proposed below. Efficient evaluation of the derivatives of the eigenpairs, using finite difference and the CA approach, is presented later.

3. Reducing the number of differential equations

Due to the linearity of Eq. (11), we can use superposition and divide it into the following 3 equations with identical initial conditions

$$\ddot{q}_k^{(i)} + 2\omega_k\zeta_k\dot{q}_k^{(i)} + \omega_k^2 q_k^{(i)} = F_k^{(i)} \quad i = 1,2,3 \tag{12}$$

where

$$F_k^{(1)} = \mathbf{\Phi}_k^{\mathrm{T}}\frac{\partial \mathbf{R}}{\partial X_j} + \frac{\partial \mathbf{\Phi}_k^{\mathrm{T}}}{\partial X_j}\mathbf{R} \quad F_k^{(2)} = -2\frac{\partial \omega_k}{\partial X_j}\zeta_k\dot{Z}_k \quad F_k^{(3)} = -\frac{\partial \omega_k^2}{\partial X_j}Z_k \tag{13}$$

$$q_k = \sum_{i=1}^{3} q_k^{(i)} \quad \dot{q}_k = \sum_{i=1}^{3}\dot{q}_k^{(i)} \quad \ddot{q}_k = \sum_{i=1}^{3}\ddot{q}_k^{(i)} \tag{14}$$

Noting that the right-hand sides of Eq. (6) and Eq. (12) for $i = 1$ are

$$P_k = \mathbf{\Phi}_k^{\mathrm{T}}\mathbf{R} \tag{15}$$

$$F_k^{(1)} = \mathbf{\Phi}_k^{\mathrm{T}}\frac{\partial \mathbf{R}}{\partial X_j} + \frac{\partial \mathbf{\Phi}_k^{\mathrm{T}}}{\partial X_j}\mathbf{R} \tag{16}$$

and assuming that the load vector can be expressed in the form $\mathbf{R}(\mathbf{X},t) = \mathbf{R}(\mathbf{X})g(t)$, then Eqs. (15) and (16) describe similar functions in time with different amplitudes. For zero initial conditions (or, if we neglect the influence of the homogeneous solution), the ratio between the two displacement functions of Eqs. (6) and (12) is equal to the ratio between the right-hand side terms. Thus, given the solutions Z_k of Eq. (6) for all p modes, the solutions $q_k^{(1)}$ of Eq. (12) for $i = 1$ can be determined directly by

$$q_k^{(1)} = Z_k\frac{\mathbf{\Phi}_k^{\mathrm{T}}\frac{\partial \mathbf{R}}{\partial X_j} + \frac{\partial \mathbf{\Phi}_k^{\mathrm{T}}}{\partial X_j}\mathbf{R}}{\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{R}} \tag{17}$$

To find $q_k^{(2)}$, $q_k^{(3)}$, Eq. (12) must be solved for $i = 2$ and $i = 3$. For X_1 we have to solve the two equations

$$\ddot{q}_k^{(2)} + 2\omega_k\zeta_k\dot{q}_k^{(2)} + \omega_k^2 q_k^{(2)} = -2\frac{\partial \omega_k}{\partial X_1}\zeta_k\dot{Z}_k \tag{18}$$

$$\ddot{q}_k^{(3)} + 2\omega_k\zeta_k\dot{q}_k^{(3)} + \omega_k^2 q_k^{(3)} = -\frac{\partial \omega_k^2}{\partial X_1}Z_k \tag{19}$$

Given the solutions of Eqs. (18) and (19) with respect to X_1, it is observed that the solutions for any other variable X_j can be determined directly by

$$q_k^{(2)}(X_j) = \frac{\frac{\partial \omega_k}{\partial X_j}}{\frac{\partial \omega_k}{\partial X_1}} q_k^{(2)}(X_1) \quad q_k^{(3)}(X_j) = \frac{\frac{\partial \omega_k^2}{\partial X_j}}{\frac{\partial \omega_k^2}{\partial X_1}} q_k^{(3)}(X_1) \tag{20}$$

In the particular case where $\mathbf{\Phi}_k$ and $\mathbf{R}$ are orthogonal we obtain $P_k = \mathbf{\Phi}_k^{\mathrm{T}}\mathbf{R} = 0$. From Eq. (6) we have $Z_k = \dot{Z}_k = \ddot{Z}_k = 0$, and from Eqs. (17) to (19) we find $q_k^{(1)} = q_k^{(2)} = q_k^{(3)} = 0$.

In summary, assuming a problem with p considered mode shapes and n design variables, the number of times that the differential equation (11) must be solved in order to perform sensitivity analysis is usually pn. Considering the procedure presented in this section and assuming that the solution of the analysis problem [Eq. (6)] is known, the number of times that the differential equations must be solved in order to perform sensitivity analysis is only $2p$ [Eqs. (18) and (19)]. Thus, the ratio between the two numbers is $pn/2p = n/2$, which means a significant reduction in the computational cost. For example, for a problem with 10 design variables, the procedure presented requires about 20% of the effort involved in complete sensitivity analysis.

4. Derivatives of the eigenpairs

4.1. Analytical derivatives

For simplicity, we eliminate the subscripts k (mode shapes) and j (design variables). Thus, the eigenproblem of Eq. (3) is expressed as

$$\mathbf{K\Phi} = \lambda \mathbf{M\Phi} \tag{21}$$

The eigenvector is often normalized such that

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M\Phi} = 1 \tag{22}$$

To evaluate the derivatives of the eigenpairs ($\partial\mathbf{\Phi}/\partial X$ and $\partial\lambda/\partial X$), we differentiate Eqs. (21) and (22) with respect to a design variable X and rearrange to obtain

$$(\mathbf{K} - \lambda\mathbf{M})\frac{\partial \mathbf{\Phi}}{\partial X} - \frac{\partial \lambda}{\partial X}\mathbf{M\Phi} = -\left(\frac{\partial \mathbf{K}}{\partial X} - \lambda\frac{\partial \mathbf{M}}{\partial X}\right)\mathbf{\Phi} \tag{23}$$

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\frac{\partial \mathbf{\Phi}}{\partial X} = -\frac{1}{2}\mathbf{\Phi}^{\mathrm{T}}\frac{\partial \mathbf{M}}{\partial X}\mathbf{\Phi} \tag{24}$$

or, in matrix form

$$\begin{bmatrix} \mathbf{K} - \lambda\mathbf{M} & -\mathbf{M\Phi} \\ \mathbf{\Phi}^{\mathrm{T}}\mathbf{M} & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial \mathbf{\Phi}}{\partial X} \\ \frac{\partial \lambda}{\partial X} \end{Bmatrix} = -\begin{Bmatrix} \left(\frac{\partial \mathbf{K}}{\partial X} - \lambda\frac{\partial \mathbf{M}}{\partial X}\right)\mathbf{\Phi} \\ \frac{1}{2}\mathbf{\Phi}^{\mathrm{T}}\frac{\partial \mathbf{M}}{\partial X}\mathbf{\Phi} \end{Bmatrix} \tag{25}$$

In the solution of Eq. (25) care must be taken because the principal minor $(\mathbf{K} - \lambda\mathbf{M})$ is singular. In many cases we are interested only in the derivatives $\partial\lambda/\partial X$. These derivatives may be obtained by premultiplying Eq. (23) by $\mathbf{\Phi}^{\mathrm{T}}$ and rearranging

$$\frac{\partial \lambda}{\partial X} = \frac{\mathbf{\Phi}^{\mathrm{T}}\left(\frac{\partial \mathbf{K}}{\partial X} - \lambda\frac{\partial \mathbf{M}}{\partial X}\right)\mathbf{\Phi}}{\mathbf{\Phi}^{\mathrm{T}}\mathbf{M\Phi}} \tag{26}$$

Note that this is only correct if the eigenvalue λ is distinct.

Several methods have been proposed to solve Eq. (25). In general, the solution involves much computational effort. Specifically, a matrix of the order $(m + 1)$, m being the number of degrees of freedom, must

be factorized for each of the p considered mode shapes. In addition, the matrices $\partial\mathbf{K}/\partial X$, $\partial\mathbf{M}/\partial X$ must be calculated and forward and backward substitutions must be carried out for each design variable.

4.2. Finite-difference derivatives

In the forward-difference method, the derivatives are approximated from the exact displacements at the original point X and at the perturbed point $X+\delta X$ by

$$\frac{\partial\mathbf{\Phi}}{\partial X}=\frac{\mathbf{\Phi}(X+\delta X)-\mathbf{\Phi}(X)}{\delta X} \tag{27}$$

where δX is a predetermined step-size. The accuracy can be improved by adopting the central-difference approximation, where the derivatives are computed from the exact displacements at the two points $X-\delta X$ and $X+\delta X$ by

$$\frac{\partial\mathbf{\Phi}}{\partial X}=\frac{\mathbf{\Phi}(X+\delta X)-\mathbf{\Phi}(X-\delta X)}{2\delta X} \tag{28}$$

Finite-difference methods are the easiest to implement and therefore they are attractive in many applications. When $\mathbf{\Phi}(X)$ is known, application of Eq. (27) involves only one additional calculation of the displacements at $X+\delta X$ whereas Eq. (28) requires calculation at the two points $X-\delta X$ and $X+\delta X$. For a problem with n design variables, finite-difference derivative calculations require repetition of the analysis for $n+1$ [Eq. (27)] or $2n+1$ [Eq. (28)] different design points. This procedure is usually not efficient compared to, for example, analytical and semi-analytical methods. An efficient solution procedure using the CA approach is described below.

As noted earlier, finite-difference approximations might have accuracy problems. The following two sources of errors should be considered whenever these approximations are used:

(a) The truncation error, which is a result of neglecting terms in the Taylor series expansion of the perturbed response.
(b) The condition error, which is the difference between the numerical evaluation of the function and its exact value. Examples for this type of error include round-off error in calculating $\partial\mathbf{\Phi}/\partial X$ from the original and perturbed values of $\mathbf{\Phi}$, and calculation of the response by approximate analysis. The latter can also be the result of a finite number of iterations being used within an iterative procedure.

These are two conflicting considerations. That is, a small step size δX will reduce the truncation error, but may increase the condition error. In some cases there may not be any step size which yields an acceptable error. Some considerations for choosing the forward-difference step-size are discussed elsewhere (Burton, 1992). In certain applications, truncation errors are not of major importance since it is often sufficient to find the average rate of change in the structural response and not necessarily the accurate local rate of change at a given point. Therefore, to eliminate round-off errors due to approximations it is recommended to increase the step-size.

It is well known that relatively small response values are not calculated as accurately as large response values (Haftka and Gurdal, 1993). The same applies to derivatives. Thus, it would be difficult to evaluate accurately small response derivatives by finite difference or other approximations. Fortunately, it is usually not important to evaluate accurately relatively small derivative values. The relative magnitude of the derivatives can be estimated from the ratio $(\partial\mathbf{\Phi}/\mathbf{\Phi})/(\partial X/X)$.

5. Efficient finite-difference derivatives

5.1. The reduced eigenproblem

Eigenproblem reanalysis by the CA method has been discussed in detail in previous studies (Kirsch, 2003b; Kirsch and Bogomolni, 2004). For completeness of presentation, the solution procedure is briefly described in this section. Given an initial design, we assume that the corresponding stiffness matrix $\mathbf{K}_0$ is given in the decomposed form

$$\mathbf{K}_0 = \mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0 \tag{29}$$

where $\mathbf{U}_0$ is an upper triangular matrix. The initial eigenpair $\mathbf{\Phi}_0$, λ_0 is obtained by solving the initial eigenproblem

$$\mathbf{K}\mathbf{\Phi}_0 = \lambda_0\mathbf{M}\mathbf{\Phi}_0 \tag{30}$$

Assume a perturbation δX in the design and corresponding changes $\delta\mathbf{K}$ in the stiffness matrix and $\delta\mathbf{M}$ in the mass matrix, respectively. The modified matrices are given by

$$\mathbf{K} = \mathbf{K}_0 + \delta\mathbf{K} \quad \mathbf{M} = \mathbf{M}_0 + \delta\mathbf{M} \tag{31}$$

The object is to estimate efficiently and accurately the requested eigenpair $\mathbf{\Phi}$, λ without solving the complete set of modified equations

$$(\mathbf{K}_0 + \delta\mathbf{K})\mathbf{\Phi} = \lambda\mathbf{M}\mathbf{\Phi} \tag{32}$$

The solution process involves the following steps.

(a) Calculate the modified matrices $\mathbf{K}$, $\mathbf{M}$ [Eq. (31)].
(b) Calculate the matrix of basis vectors $\mathbf{r}_{\mathrm{B}}$

$$\mathbf{r}_{\mathrm{B}} = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s] \tag{33}$$

where $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s$ are the basis vectors, and s is much smaller than the number of degrees of freedom. For any requested eigenpair $\mathbf{\Phi}$, λ the basis vectors are determined separately, using the steps described in the next section.
(c) Calculate the reduced matrices $\mathbf{K}_{\mathrm{R}}$ and $\mathbf{M}_{\mathrm{R}}$ by

$$\mathbf{K}_{\mathrm{R}} = \mathbf{r}_{\mathrm{B}}^{\mathrm{T}}\mathbf{K}\mathbf{r}_{\mathrm{B}} \quad \mathbf{M}_{\mathrm{R}} = \mathbf{r}_{\mathrm{B}}^{\mathrm{T}}\mathbf{M}\mathbf{r}_{\mathrm{B}} \tag{34}$$

(d) Solve the reduced $s \times s$ eigenproblem for the first eigenpair λ_1, $\mathbf{y}_1$

$$\mathbf{K}_{\mathrm{R}}\mathbf{y}_1 = \lambda_1\mathbf{M}_{\mathrm{R}}\mathbf{y}_1 \tag{35}$$

where $\mathbf{y}_1^{\mathrm{T}}$ is a vector of unknown coefficients

$$\mathbf{y}_1^{\mathrm{T}} = \{y_1, y_2, \ldots, y_s\} \tag{36}$$

Various methods (e.g. inverse vector iteration) can be used for this purpose.
(e) Evaluate the requested mode shape $\mathbf{\Phi}$ by

$$\mathbf{\Phi} = y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + \cdots + y_s\mathbf{r}_s = \mathbf{r}_{\mathrm{B}}\mathbf{y}_1 \tag{37}$$

The requested eigenvalue is already given from Eq. (35) $\lambda = \lambda_1$.

It was found that high accuracy is often achieved with a very small number of basis vectors. In such cases the above solution procedure is most effective.

5.2. Improved basis vectors

The effectiveness of the solution approach depends, to a great extent, on the appropriate choice of the basis vectors. Proper selection of the basis vectors is perhaps the most important factor affecting the successful application of the method. It was found that the basis vectors determined by the method described in this section provide accurate results with a small computational effort.

The binomial series terms. The basis vectors for any requested eigenpair $\mathbf{\Phi}$, λ, are first calculated by the terms of the binomial series as follows (Kirsch et al., submitted for publication-b). The first basis vector is selected as

$$\mathbf{r}_1 = \mathbf{K}_0^{-1}\mathbf{M}\mathbf{\Phi}_0 \tag{38}$$

Additional vectors are calculated by the terms of the binomial series

$$\mathbf{r}_k = -\mathbf{B}\mathbf{r}_{k-1} \tag{39}$$

where matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \mathbf{K}_0^{-1}\delta\mathbf{K} \tag{40}$$

Calculation of each basis vector by Eq. (39) involves only forward and backward substitutions, since $\mathbf{K}_0$ is given in the decomposed form of Eq. (29) from the initial analysis.

Reduction of truncation errors. Substituting Eq. (40) into Eq. (39) yields

$$\mathbf{r}_k = -\mathbf{K}_0^{-1}\delta\mathbf{K}\mathbf{r}_{k-1} \tag{41}$$

It was found (Barthelemy et al., 1988; Pedersen et al., 1989) that the expression of Eq. (41) might cause inaccurate results in calculating sensitivities with respect to shape design variables. To improve the accuracy, it is possible to use the central-difference expression

$$\delta\overline{\mathbf{K}} = \mathbf{K}(X + \delta X) - \mathbf{K}(X \ - \delta X) \tag{42}$$

in Eq. (40), instead of the forward-difference expression [Eq. (31)]

$$\delta\mathbf{K} = \mathbf{K}(X + \delta X) - \mathbf{K}_0 \tag{43}$$

This modification may reduce significantly the number of basis vectors required to achieve sufficiently accurate results. In summary, the resulting expressions for calculating the basis vectors [instead of Eqs. (38)–(40)] are

$$\bar{\mathbf{r}}_1 = \mathbf{r}_1 = \mathbf{K}_0^{-1}\mathbf{M}\mathbf{\Phi}_0 \tag{44}$$

$$\bar{\mathbf{r}}_k = -\overline{\mathbf{B}}\bar{\mathbf{r}}_{k-1} \tag{45}$$

$$\overline{\mathbf{B}} = \mathbf{K}_0^{-1}\delta\overline{\mathbf{K}} \tag{46}$$

It should be noted that forward-difference derivatives [only one additional reanalysis for $(X + \delta X)$] can be used with the central difference expressions of Eqs. (44)–(46).

Gram–Schmidt orthogonalizations. To improve the accuracy of the results for the higher mode shapes, we use Gram–Schmidt orthogonalizations of the approximate mode shapes. Assume for example that we have calculated the first p eigenvectors $\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_p$ and that we want to $\mathbf{M}$-orthogonalize $\mathbf{\Phi}_{p+1}$ to these eigenvectors. For this purpose, we $\mathbf{M}$-orthogonalize the basis vectors $\bar{\mathbf{r}}_k$ of $\mathbf{\Phi}_{p+1}$ to the lower eigenvectors by

$$\mathbf{r}_k = \bar{\mathbf{r}}_k - \sum_{i=1}^{p}(\mathbf{\Phi}_i^{\mathrm{T}}\mathbf{M}\bar{\mathbf{r}}_k)\mathbf{\Phi}_i \tag{47}$$

This expression is further developed below.

It was found that in many cases the basis vectors determined by Eqs. (44) and (45) come close to being linearly-dependent. As a result, numerical errors might occur. To overcome this difficulty, Gram–Schmidt orthogonalizations are used to generate a new set of orthogonal basis vectors $\mathbf{V}_k$ $(k = 1, 2, \ldots, s)$. The advantage is that more accurate results are obtained with the new vectors that satisfy the conditions $\mathbf{V}_k^{\mathrm{T}}\mathbf{M}\mathbf{V}_j = \delta_{kj}$. The first normalized basis vector $\mathbf{V}_1$ is determined by

$$\mathbf{V}_1 = |\mathbf{r}_1^{\mathrm{T}}\mathbf{M}\mathbf{r}_1|^{-1/2}\mathbf{r}_1 \tag{48}$$

Additional basis vectors $(k = 2, \ldots, s)$ are generated by the following expression for the non-normalized vectors $\overline{\mathbf{V}}_k$

$$\overline{\mathbf{V}}_k = \mathbf{r}_k - \sum_{j=1}^{k-1}(\mathbf{r}_k^{\mathrm{T}}\mathbf{M}\mathbf{V}_j)\mathbf{V}_j \tag{49}$$

Eq. (49) is used for the first mode shape. For the higher modes we use also Gram–Schmidt orthogonalizations of the modes [Eq. (47)] and obtain for the $(p+1)$th mode

$$\overline{\mathbf{V}}_k = \mathbf{r}_k - \sum_{j=1}^{k-1}(\mathbf{r}_k^{\mathrm{T}}\mathbf{M}\mathbf{V}_j)\mathbf{V}_j - \sum_{i=1}^{p}(\mathbf{\Phi}_i^{\mathrm{T}}\mathbf{M}\mathbf{r}_k)\mathbf{\Phi}_i \tag{50}$$

The non-normalized vectors $\overline{\mathbf{V}}_k$, calculated by Eq. (49) or Eq. (50), are normalized by

$$\mathbf{V}_k = |\overline{\mathbf{V}}_k^{\mathrm{T}}\mathbf{M}\overline{V}_k|^{-1/2}\overline{\mathbf{V}}_k \tag{51}$$

6. Numerical examples

6.1. Cantilever column

All dimensions in this example are arbitrary. The column shown in Fig. 1 of a length $L = 100$ consists of n equally sized beam elements. The uniform bending stiffness is $EI = 10^9$, the distributed mass is 10^3, and the length $X = L/n$ of the elements is taken as the design variable. The structure is subjected to the loading

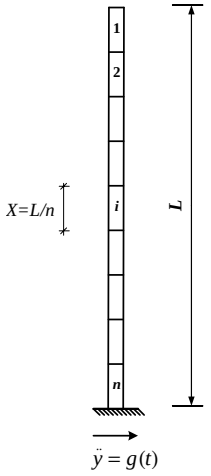

Fig. 1. Column example.

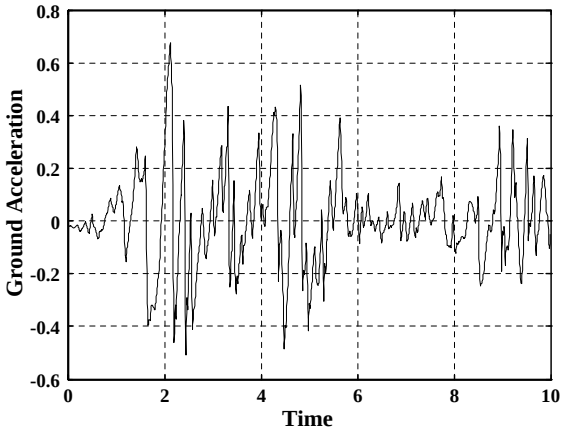

Fig. 2. El Centro earthquake.

(ground acceleration) of the El Centro earthquake shown in Fig. 2, normalized to be at the 10% range of all earthquakes expected to appear in 50 years (Somerville et al., 1997). The object is to evaluate the sensitivity of the horizontal displacement at the top of the column with respect to X.

It has been shown (Pedersen et al., 1989) that the sensitivity errors according to the traditional semi-analytical method are proportional to ηn^2, where $\eta = \delta X/X$ denotes the relative perturbation of the design variable, and δX is the perturbation. Derivatives of the eigenvalues, $\partial\lambda_i/\partial X$, are calculated by the following methods:

$\partial\lambda_i/\partial X(\mathrm{FD})$ = forward-difference derivatives using exact analysis formulation.
$\partial\lambda_i/\partial X(\mathrm{CA}s)$ = forward-difference derivatives using the CA method with s basis vectors.

The percentage errors $E(\mathrm{d}\lambda_i) = 100[\partial\lambda_i/\partial X(\mathrm{FD}) - \partial\lambda_i/\partial X(\mathrm{CA}s)]/[\partial\lambda_i/\partial X(\mathrm{FD})]$ were calculated for the first five mode shapes. Results obtained with various numbers of basis vectors for different numbers of column elements n ($n = 50$, 100, 200, 300), and $\eta = \delta X/X = 0.01$, 0.001, 0.0001 (using the forward-difference $\delta\mathbf{K}$ and the central-difference $\delta\overline{\mathbf{K}}$) are shown in Tables 1 and 2. It is observed that the results achieved by the CA method are very close to those obtained by exact analysis formulation. Using central-difference,

Table 1
Errors in eigenvalues, cantilever column, forward difference $\delta\mathbf{K}$

n	η	CAs	$E(\mathrm{d}\lambda_1)$	$E(\mathrm{d}\lambda_2)$	$E(\mathrm{d}\lambda_3)$	$E(\mathrm{d}\lambda_4)$	$E(\mathrm{d}\lambda_5)$
50	0.01	CA5	0.00304	0.00018	1.4e−05	3.6e−06	1.6e−07
	0.001	CA2	0.21313	0.00336	4.6e−05	7.8e−05	3.3e−05
	0.0001	CA2	0.00205	0.00026	2.5e−05	1.2e−05	2.65e−07
100	0.01	CA6	0.14660	0.00396	7.8e−06	1e−05	2e−06
	0.001	CA3	0.01979	0.00061	4e−05	1.5e−05	2.7e−06
	0.0001	CA2	0.01977	0.00017	0.00055	0.00011	4.8e−05
200	0.01	CA10	0.03720	0.00010	7.2e−05	4.2e−05	5.1e−06
	0.001	CA4	0.01553	0.00014	0.00020	0.00011	0.00010
	0.0001	CA3	0.05068	0.04500	0.00729	0.00086	0.00065
300	0.01	CA13	0.06554	0.00164	3.3e−05	0.00531	0.00308
	0.001	CA5	0.14591	0.00400	0.00179	0.00120	7.3e−05
	0.0001	CA3	1.64950	0.11213	0.01491	0.00421	0.00509

Table 2
Errors in eigenvalues, cantilever column, central difference $\delta\overline{\mathbf{K}}$

n	η	CAs	$E(\mathrm{d}\lambda_1)$	$E(\mathrm{d}\lambda_2)$	$E(\mathrm{d}\lambda_3)$	$E(\mathrm{d}\lambda_4)$	$E(\mathrm{d}\lambda_5)$
50	0.01	CA2	0.21378	0.13186	0.01121	0.00691	0.00268
	0.001	CA2	0.00014	2.6e−05	8e−07	9.6e−07	2.9e−05
	0.0001	CA2	0.00226	0.00026	2e−05	1.2e−06	2.6e−06
100	0.01	CA3	0.00105	0.00119	0.00074	0.00015	0.00106
	0.001	CA2	0.00096	0.00035	2e−05	1.2e−05	3.6e−06
	0.0001	CA2	0.00226	0.00026	4e−05	1.5e−05	2.7e−06
200	0.01	CA4	0.00182	0.00016	0.00038	0.00010	0.00020
	0.001	CA2	0.01507	0.00015	0.00020	0.00011	0.00010
	0.0001	CA2	0.05069	0.04494	0.00730	0.00086	0.00064
300	0.01	CA4	0.06166	0.07698	0.20689	0.15456	0.01501
	0.001	CA2	0.14835	0.00465	0.00184	0.00118	7.8e−05
	0.0001	CA2	1.64930	0.11247	0.01491	0.00422	0.00509

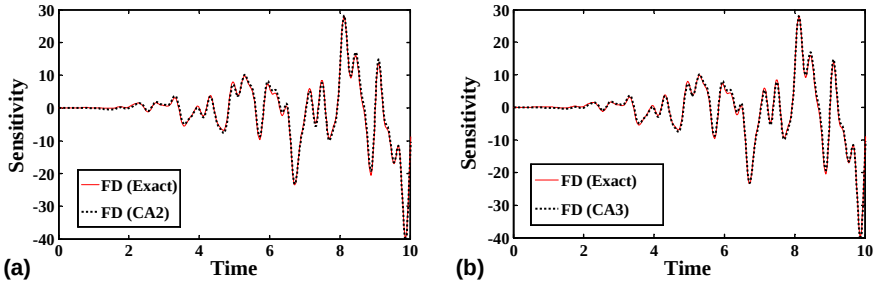

Fig. 3. Displacement sensitivities: (a) $n = 100$, $\eta = 0.0001$; (b) $n = 300$, $\eta = 0.0001$.

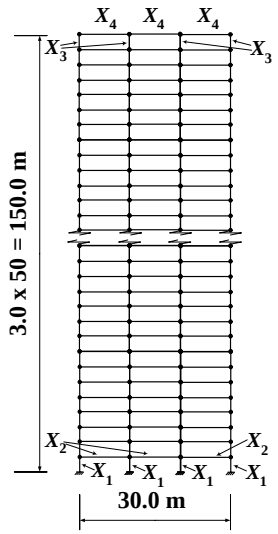

Fig. 4. Fifty-story frame.

a very small number of basis vectors is sufficient to obtain accurate results. In general, the number of basis vectors needed to obtain small errors is increased with n and with η. However, in some cases, larger errors are obtained for smaller perturbations due to condition errors. In any case, all the errors shown in Table 2 are very small. The displacement sensitivities for the two cases $n = 100$, $\eta = 0.0001$ and $n = 300$, $\eta = 0.0001$, shown in Fig. 3 for the forward-difference $\delta\mathbf{K}$ demonstrate the high accuracy achieved by the CA method.

6.2. Fifty-story frame

Consider the fifty-story frame shown in Fig. 4. The number of degrees of freedom is 600, and the damping ratios for all modes are 0.05. The masses are assumed to be concentrated at the joints, and only horizontal inertia forces are considered. The inertia force is due to the frame self-weight and an additional concentrated mass of 50 ton in an internal joint and 25 ton in an external joint. The width of all elements is 0.5 m, the depth of all columns is 1.0 m and the depth of all beams is 0.8 m. The modulus of elasticity is 3×10^7 kN m^2. The loading is due to the ground acceleration of the El Centro earthquake. The object is to

Table 3
Eigenvalue sensitivities, fifty-story frame

Sensitivity	Mode	FD(exact)	FD(CA2)
$\partial\lambda/\partial X_1$	1	0.0480	0.0480
	2	0.4723	0.4723
	3	1.4404	1.4404
	4	2.8998	2.8998
	5	4.978	4.978
	6	7.705	7.705
	7	11.232	11.232
	8	15.617	15.617
$\partial\lambda/\partial X_2$	1	0.0330	0.0330
	2	0.5834	0.5834
	3	1.7403	1.7403
	4	3.5348	3.5348
	5	5.9373	5.9373
	6	9.0852	9.0852
	7	12.876	12.876
	8	17.485	17.485
$\partial\lambda/\partial X_3$	1	−0.00315	−0.00315
	2	−0.02443	−0.02443
	3	−0.05199	−0.05198
	4	−0.04155	−0.04154
	5	0.06455	0.06457
	6	0.33255	0.33257
	7	0.85281	0.85282
	8	1.69010	1.69010
$\partial\lambda/\partial X_4$	1	−0.00774	−0.00779
	2	−0.06148	−0.06149
	3	−0.13726	−0.13723
	4	−0.12572	−0.12572
	5	0.13424	0.13425
	6	0.84037	0.84042
	7	2.2883	2.2884
	8	4.7361	4.7361

evaluate the sensitivities of the horizontal displacements at the 1st story and the 50th story with respect to the following four design variables;

X_1 depth of the columns in the 1st story.
X_2 depth of the beams in the 1st story.
X_3 depth of the columns in the 50th story.
X_4 depth of the beams in the 50th story.

Choosing the time-step $\Delta t = 0.02$ s and considering the first 8 mode shapes, the results obtained by forward-difference derivatives using exact analysis formulation [FD (exact)] are compared with those achieved by the CA approach with only 2 basis vectors [FD (CA2)]. Table 3 shows the eigenvalue sensitivities, Fig. 5 shows the displacements, and Figs. 6 and 7 show the displacement sensitivities of the 1st and the 50th stories. It is observed that high accuracy is achieved by the procedure presented.

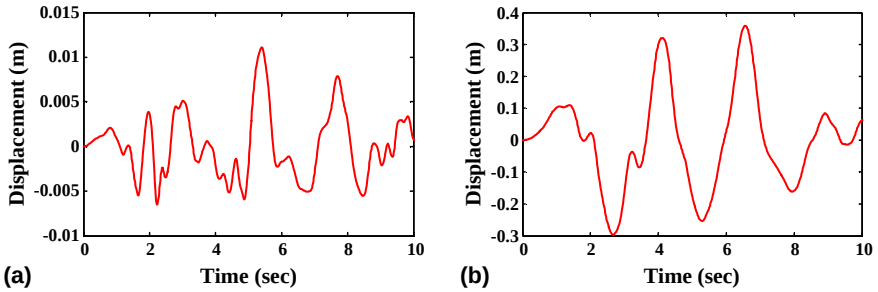

Fig. 5. Horizontal displacements: (a) 1st floor, (b) 50th floor.

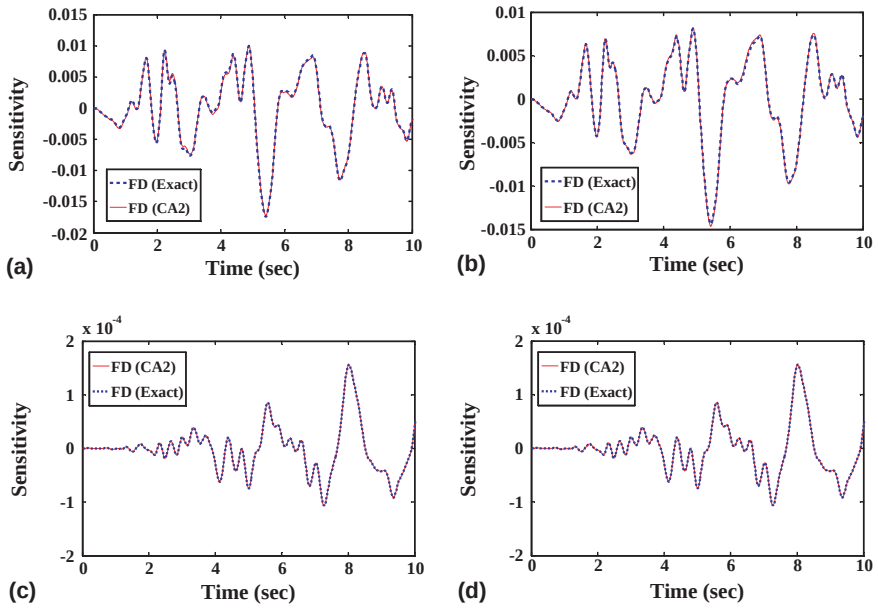

Fig. 6. Displacement sensitivities, 1st floor, with respect to: (a) X_1, (b) X_2, (c) X_3, (d) X_4.

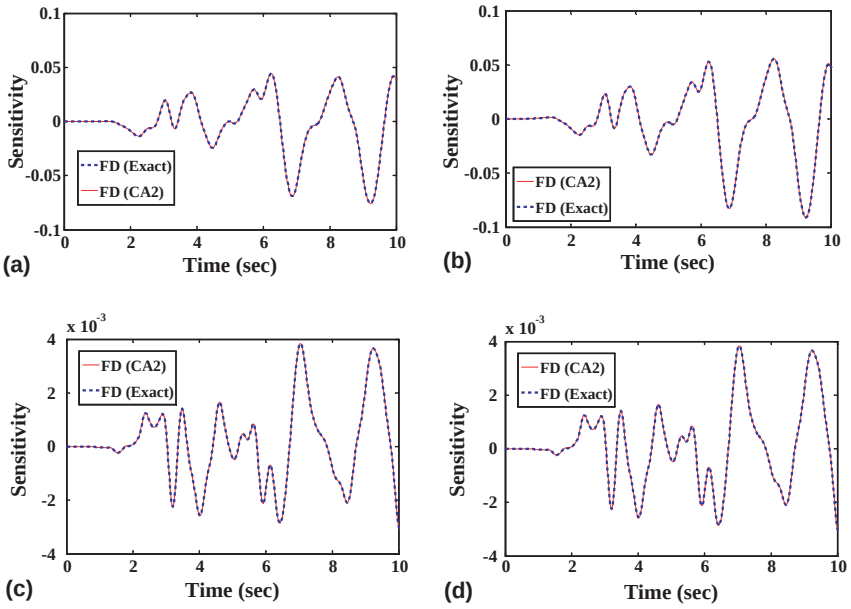

Fig. 7. Displacement sensitivities, 50st floor, with respect to: (a) X_1, (b) X_2, (c) X_3, (d) X_4.

Solving various frames with different numbers of degrees of freedom, it was found that in all cases only 2 basis vectors provide accurate sensitivities. This result is typical for small perturbations in a single design variable.

7. Conclusions

Calculation of response derivatives with respect to design variables often involves much computational effort, particularly in large structural systems subjected to dynamic loading. Approximation concepts, which are often used to reduce the computational cost involved in repeated analysis, are usually not sufficiently accurate for sensitivity analysis.

In this study efficient sensitivity analysis, using the recently developed combined approximations approach and finite differences, is presented. Assuming modal analysis, a procedure intended to reduce the number of differential equations that must be solved during the solution process is proposed. Computational procedures intended to improve the accuracy of the approximations are developed, and efficient evaluation of the response derivatives by the combined approximations approach is presented. Numerical examples show that accurate results can be achieved efficiently. In general, sensitivity analysis by the CA method is used in problems of small perturbations in a single design variable. In such cases a very small number of basis vectors provide accurate results even for structures having large numbers of degrees of freedom.

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